De Jonquieres' formula for families of nodal curves

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Outline

1. Introduction
   - Motivation
2. Hilbert Schemes
   - Tautological Modules
   - Transfer theorems
3. de Jonquieres' formula
   - for a curve
   - for a family of nodal curves

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- By Riemann-Roch, $D \in X^{(d)}$ satisfies $dim|D| \geq r$ iff

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- As D varies, we get a morphism of vector bundles

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- At $D \in X^{(d)}$, $\Lambda_d K_X|_D = H^0(D, K_X|_D)$. Since p is flat, $\Lambda_d K_X$ is a vector bundle of rank d.

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- By Porteous' formula $C_d^r = \triangle_{g-d+r,r}(c_t(\Lambda_d K_X))$.
- If this class is not zero, then $C_d^r \neq \emptyset$, i.e. $\exists$ a special divisor.

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- For a d-partition $\mu = (n_1, \cdots, n_k)$, the r-degeneracy locus on a diagonal locus $\Gamma_\mu \subset X^{(d)}$, i.e. $\triangle_{d-r,1}(c_t(\Lambda_d L))|_{\Gamma_\mu} = c_{d-r}(\Lambda_d L)|_{\Gamma_\mu}$ is the answer to the question provided this is finite, i.e. $k = d - r$.

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- Define $\Gamma^{(m)} := \frac{1}{2} c_m^{-1}(D^{(m)})$.

Motivation

- For enumerative geometry of Hilbert schemes one studies flag Hilbert schemes, $X_B^{[m,m-1]}$, parametrizing (z_1, z_2), where $z_1 \supset z_2$ and $z_1 \subset X_b$ for some b.(e.g. Lehn, Gottsche)

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On $X_B^{[m,m-1]}$, we have

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- Need INTERSECTION CALCULUS with $\Gamma^{(m)}$ and TRANSFER from $X_B^{[m-1]}$ to $X_B^{[m]}$.

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$$\begin{array}{ccc} F_j^{n,m}(\theta) & \to & X_B^{[m]} \\ \downarrow & & \\ (X_T^\theta)^{[m-n]}. & & \end{array}$$

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- e.g. $\Gamma^{(m)}\Gamma_{(m)} = \sum_{\theta,i} \frac{i(m-i)m}{2} C_i^m(\theta) - \binom{m}{2}\Gamma_{(m)}[\omega]$.

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3. $\tau_{m,f}((-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta)[\alpha_.]\beta_{(m)}) =$

$$\theta^*(\beta)F^{n+1,m}_{j,\nu}(\theta)[\alpha.] + (-\Gamma^{(m)})F^{n,m}_{j,\nu+1_1}(\theta)[\tau_{m-n,X_T^\theta}(\alpha.\beta|_{X_T^\theta})] -$$
$$F^{n,m}_{j,\nu+1_1}(\theta)[e^{n,m}_{j+1}(\tau_{m-n,X_T^\theta}(\alpha.\beta|_{X_T^\theta}))] +$$
$$F^{n,m}_{j,\nu+1_1}(\theta)[\tau_{m-n,X_T^\theta}(e^{n,m-1}_{j+1}(\alpha.)\beta|_{X_T^\theta})].$$

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If a_k is a multiplicity of node,

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5. $\tau_{m,p}((-\Gamma^{(m-1)})F_{j,\nu}^{n,m-1}(\theta)) =$
$$(-\Gamma^{(m)})F_{j+1,\nu}^{n+1,m}(\theta) - F_{j+1,\nu}^{n+1,m}(\theta)[(\tfrac{n-j-1}{n}\psi_{j+2}^n + \tfrac{j+1}{n}\psi_{j+1}^n)] + \tfrac{n-j}{n-1}F_{j,\nu}^{n+1,m}(\theta)[\psi_j^{n-1}\alpha.] + \tfrac{j+1}{n-1}F_{j+1,\nu}^{n+1,m}(\theta)[\psi_j^{n-1}\alpha.].$$

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- $k = 2$ and $a_1 + a_2 = m$,

$$\begin{aligned} c_2(\Lambda_m(L)|_{\Gamma_{(a_1,a_2)}}) =& a_1 a_2 L^2 + (a_1\binom{a_2}{2} + a_2\binom{a_1}{2})L\omega \\ & - a_1(a_1 a_2 + 2\binom{a_2}{2})L + \binom{a_1}{2}\binom{a_2}{2}\omega^2 \\ & - (a_1 a_2\binom{a_1}{2} + a_1\binom{a_2}{3} + a_1^2\binom{a_2}{2} \\ & + a_1\binom{a_2}{2}\frac{2a_2-1}{3})\omega. \end{aligned}$$

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Lemma

$$P^{r+1}(L) \cong \Lambda_{r+1}(L)|_{\Gamma_{(r+1)}}$$

,where $P^{r+1}(L) := \pi_{1*}(\pi_2^* L \otimes \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^{r+1})$ is the bundle of principal parts or Jet bundle of L.

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- If $k = 1, L = K$ canonical sheaf, then the number of *Weierstrass points* is $c_1(\Lambda_g(K)|_{\Gamma_g}) = (g-1)g(g+1)$.

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- For the de Jonquieres' problem, we consider a line bundle L on X, a vector bundle E on B such that $E \subset \pi_* L$.
  That is, we have a family of g^r_d's.
  For $b \in B$, $E_b \subset H^0(X_b, L_b)$.

de Jonquieres' formula for a family

- Consider 1-parameter family, i.e. $dimB = 1$. We have only finitely many nodes.
- For the de Jonquieres' problem, we consider a line bundle L on X, a vector bundle E on B such that $E \subset \pi_* L$.
  That is, we have a family of g^r_d's.
  For $b \in B$, $E_b \subset H^0(X_b, L_b)$.
- Compute the degeneracy locus of

$$\phi : \pi^{[m]*}(E) \to \Lambda_m(L).$$

de Jonquieres' formula for a family cont.

By Porteous formula, we need to compute

$$c_2(\Lambda_m L - \pi^{[m]*}E)|_{\Gamma_{(m)}} = c_2(\Lambda_m L|_{\Gamma_{(m)}}) - c_1(\Lambda_m L|_{\Gamma_{(m)}})x,$$

$$c_3(\Lambda_m L - \pi^{[m]*}E)|_{\Gamma_{(a_1,a_2)}} = c_3(\Lambda_m L|_{\Gamma_{(a_1,a_2)}}) - c_2(\Lambda_m L|_{\Gamma_{(a_1,a_2)}})y,$$

where $x = (\pi^{[m]})^* c_1(E) \cap [\Gamma_{(m)}]$ and $y = (\pi^{[m]})^* c_1(E) \cap [\Gamma_{(a_1,a_2)}]$.

For 1-parameter family of nodal curves and for a single block, de Jonquieres' formula is

$$c_2(\Lambda_m L - (\pi^{[m]})^* E) \cap [\Gamma_{(m)}] = c_2(\Lambda_m L|_{\Gamma_{(m)}}) - c_1(\Lambda_m L|_{\Gamma_{(m)}})x$$
$$= \binom{m}{2} L^2 + (m-1)\binom{m}{2} L\omega + (3\binom{m+1}{4} - \binom{m}{3})\omega^2 - \binom{m+1}{4}\sigma$$
$$- (mL + \binom{m}{2}\omega - \sigma \sum_{i=1}^{m-1} \frac{i(m-i)m}{2} C_i^m)x,$$

where σ is number of nodes and $x = (\pi^{[m]})^* c_1(E) \cap [\Gamma_{(m)}]$.

Thank you.